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Lectures on the Theory of Reciprocants.

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[Reported by J. Hammond.]

LECTURE XI.

We may write for the Annihilator of an Invariant

$$\Omega = a_0\dot{a}_1 + 2a_1\dot{a}_2 + 3a_2\dot{a}_3 + \ldots . ja_{j-1}\dot{a}_j$$

and for its opposite

$$O = ja_1\dot{a}_0 + (j-1)\,a_2\dot{a}_1 + (j-2)\,a_3\dot{a}_2 + \ldots . + a_j\dot{a}_{j-1},$$

where the pointed letters $\dot{a}_0$, $\dot{a}_1$, $\dot{a}_2$, ... $\dot{a}_j$ stand for the partial differential operators

$$\partial_{a_0},\ \partial_{a_1},\ \partial_{a_2},\ \ldots .\ \partial_{aj}.$$

Suppose Ω and O to operate on any function $U(a_0, a_1, a_2, \ldots . a_j)$; then

$$\Omega O U = (\Omega . O + \Omega * O)\, U$$

and

$$O\Omega U = (O . \Omega + O * \Omega)\, U,$$

where the full stop between O and Ω signifies multiplication, and the asterisk operation on the unpointed letters only. Thus,

$$\Omega . O = O . \Omega,$$

and, consequently, $(\Omega O - O\Omega)\, U = (\Omega * O - O * \Omega)\, U.$

Now, $\Omega * OU = \{1 . ja_0\dot{a}_0 + 2\,(j-1)\,a_1\dot{a}_1 + 3\,(j-2)\,a_2\dot{a}_2 + \ldots . + j . 1a_{j-1}\dot{a}_{j-1}\}\, U,$

and $O * \Omega U = \{1 . ja_1\dot{a}_1 + 2\,(j-1)\,a_2\dot{a}_2 + \ldots . + (j-1)\, 2a_{j-1}\dot{a}_{j-1} + j . 1a_j\dot{a}_j\}\, U,$

whence we readily obtain

$$\begin{aligned}(\Omega O - O\Omega)\, U = {} & j\,(a_0\dot{a}_0 + a_1\dot{a}_1 + a_2\dot{a}_2 + \ldots . + a_j\dot{a}_j)\, U \\ & -2\,(a_1\dot{a}_1 + 2a_2\dot{a}_2 + 3a_3\dot{a}_3 + \ldots . + ja_j\dot{a}_j)\, U.\end{aligned}$$

Introducing the conditions of homogeneity and isobarism, viz.

$$(a_0\dot{a}_0 + a_1\dot{a}_1 + a_2\dot{a}_2 + \ldots . + a_j\dot{a}_j) \quad U = iU$$

and

$$(a_1\dot{a}_1 + 2a_2\dot{a}_2 + 3a_3\dot{a}_3 + \ldots . + ja_j\dot{a}_j)\, U = wU,$$

where i and w denote the degree and weight of U, supposed now to be a rational integral homogeneous and isobaric function (or, to avoid a tedious periphrasis, say a *gradient*), we see that if the complete type of the gradient U is w; i, j,

$$(\Omega O - O\Omega)\, U = (ij - 2w)\, U = \eta U,$$

where η is the excess.

Since the operation of O increases the weight of the operand by unity, but does not alter either its degree or its extent, it is clear that the type of $O^\theta U$ is $w + \theta$; i, j. The excess of $O^\theta U$ is therefore

$$ij - 2(w + \theta) = \eta - 2\theta,$$

and the theorem just proved shows that

$$(\Omega O - O\Omega)\, O^\theta U = (\eta - 2\theta)\, O^\theta U.$$

From this we pass on to prove that $\Omega O^q - O^q\Omega$, acting on any gradient as its objective, is equivalent to $q(\eta - q + 1)\, O^{q-1}$; *i. e.* when q is any positive integer we shall show that

$$(\Omega O^q - O^q\Omega)\, U = q(\eta - q + 1)\, O^{q-1}U.$$

The subsequent consideration of a special case of this formula, in which U is replaced by any invariant I, will enable us to prove that there can be no invariants for which the excess $ij - 2w$ is negative. Let

$$O^{q-\theta}\Omega O^\theta U = P_\theta U;$$

then $$O^{q-\theta-1}\Omega O^{\theta+1}U = P_{\theta+1}U,$$

and therefore $$(P_{\theta+1} - P_\theta)\, U = O^{q-\theta-1}(\Omega O - O\Omega)\, O^\theta U.$$

Substituting in this for

$$(\Omega O - O\Omega)\, O^\theta U \text{ its value } (\eta - 2\theta)\, O^\theta U,$$

we have $$(P_{\theta+1} - P_\theta)\, U = (\eta - 2\theta)\, O^{q-1}U.$$

Hence

$$\begin{aligned}(P_q - P_0)\, U &= \{(P_1 - P_0) + (P_2 - P_1) + (P_3 - P_2) + \ldots\ldots + (P_q - P_{q-1})\}\, U\\ &= \{\eta + (\eta - 2) + (\eta - 4) + \ldots\ldots + (\eta - 2q + 2)\}\, O^{q-1}U\\ &= q(\eta - q + 1)\, O^{q-1}U.\end{aligned}$$

But since $P_q = \Omega O^q$ and $P_0 = O^q\Omega$, this result may be written

$$(\Omega O^q - O^q\Omega)\, U = q(\eta - q + 1)\, O^{q-1}U.$$

If now $U = I$, an invariant, we have $\Omega U = 0$, and our formula becomes

$$\Omega O^q I = q(\eta - q + 1)\, O^{q-1}I.$$

Writing in succession $q = m,\ m - 1, \ldots\ldots 1$, we obtain

$$\begin{aligned} m(\eta - m + 1)\, O^{m-1}I &= \Omega O^m I\\ (m - 1)(\eta - m + 2)\, O^{m-2} &= \Omega O^{m-1}I\\ (m - 2)(\eta - m + 3)\, O^{m-3}I &= \Omega O^{m-2}I\\ \ldots\ldots\ldots\ldots&\ldots\ldots\\ 1\,.\,\eta I &= \Omega O I.\end{aligned}$$

By assigning to m a sufficiently large value we are able to make $O^m I$ vanish as well as ΩI; for, the type of I being $w; i, j$, that of $O^m I$ is $w + m; i, j$. But it is evident that no *gradient* can have a greater weight than ij, the product of its degree and extent, for each term is a product of i letters none of them having a weight greater than j. If, then, we suppose that $m = ij - w + 1$, the weight of $O^m I$ is

$$w + m = ij + 1.$$

Therefore

$$O^m I = 0.$$

Again,

$$\eta - m + 1 = ij - 2w - (ij - w + 1) + 1 = -w.$$

If, then, η is negative, every term in the series

$$m(\eta - m + 1),\ (m - 1)(\eta - m + 2), \ldots \ldots 2(\eta - 1),\ 1 . \eta$$

is negative and can never vanish. Hence we have successively

$$O^{m-1} I = 0,\ O^{m-2} I = 0, \ldots \ldots I = 0;$$

i. e. when $ij - 2w < 0$ no invariant of the type $w; i, j$ exists.

Observe that the elenchus of the demonstration consists in the fact that the successive numerical factors $\eta - m + 1, \eta - m + 2, \eta - m + 3, \ldots \ldots \eta$ are all non-zero on account of η being negative; but if η were positive we should eventually come to a factor $\eta - \mu$ which would be zero, and we could not conclude from $(\mu + 1)(\eta - \mu) O^\mu I$ being zero that $O^\mu I = 0$. Since $\eta - (m - 1)$ passes from $\eta - (ij - w)$ to η, *i. e.* from $-w$ to η, it passes through zero when η is positive.

The second part of Cayley's completed theorem remains to be proved, viz. that when $ij - 2w => 0$, the number of linearly independent invariants of the type $w; i, j$ is precisely equal to $\Delta(w; i, j)$; *i. e.* to $(w; i, j) - (w - 1; i, j)$. I show this by proving that if $D(w; i, j)$ is the number in question, keeping i and j constant and taking $w <= \frac{ij}{2}$,

$$D(w; i, j) + D(w - 1; i, j) + D(w - 2; i, j) + \ldots \ldots + D(0; i, j)$$

cannot be greater than

$$\Delta(w; i, j) + \Delta(w - 1; i, j) + \Delta(w - 2; i, j) + \ldots \ldots + \Delta(0; i, j),$$

and consequently, since we know that no single $D(w; i, j)$ can possibly be less than the corresponding $\Delta(w; i, j)$, it follows that

$$D(w; i, j) + D(w - 1; i, j) + D(w - 2; i, j) + \ldots \ldots + D(0; i, j)$$
$$= \Delta(w; i, j) + \Delta(w - 1; i, j) + \Delta(w - 2; i, j) + \ldots \ldots + \Delta(0; i, j);$$

and, furthermore, that each

$$D(w; i, j) = \Delta(w; i, j).$$

For if any D were greater than its corresponding Δ, some other D would have to be less, which is impossible.

This principle of reasoning may be illustrated by imagining a row of ballot-boxes and supposing it to be ascertained that no single box contains fewer white balls than black ones. If, then, there are not more white than black balls altogether, the total number of whites must be the same as that of the blacks. And since there are just as many whites as blacks distributed among the ballot-boxes, the number of white and black balls must be the same in each box; for otherwise some box must contain fewer whites than blacks, which is contrary to the hypothesis.

Observe that the sum of the Δ's is $(w;\, i,\, j)$; for

$$(w;i,j)-(w-1;i,j)+(w-1;i,j)-(w-2;i,j)+\ldots\ldots+(0;i,j)-(-1;i,j)$$
$$=(w;\, i,\, j)-(-1;\, i,\, j)$$

and

$$(-1;\, i,\, j)=0,$$

since there is no way of composing -1 with parts $0,\ 1,\ 2,\, \ldots\ldots j$. Hence what I have to show is that

$$D(w;\, i,\, j)+D(w-1;\ i,\, j)+\ldots\ldots+D(1;\, i,\, j)+D(0;\, i,\, j)=(w;\, i,\, j).$$

I want preliminarily to express $\Omega^q O^q I$ as a multiple of I.*

This can be done by a formula previously demonstrated, viz.

$$\Omega O^q I = q(\eta-q+1)\, O^{q-1} I,$$

which gives

$$\Omega^2 O^q I = q(\eta-q+1)\, \Omega O^{q-1} I = q(\eta-q+1)(q-1)(\eta-q+2)\, O^{q-2} I;$$

similarly

$$\Omega^3 O^q I = q(\eta-q+1)(q-1)(\eta-q+2)(q-2)(\eta-q+3)\, O^{q-3} I;$$

and finally, changing the order of the numerical factors,

$$\Omega^q O^q I = 1.2.3\ \ldots\ldots\ q\, \{\eta(\eta-1)(\eta-2)\ \ldots\ldots\ (\eta-q+1)\}\, I.$$

This shows that $\Omega^q O^q I$ and *à fortiori* $O^q I$ can never vanish unless $\eta-q+1$ becomes negative.

Suppose now that I^q means an invariant of the type $w-q;\, i,\, j$; its excess is $ij-2(w-q)$, and consequently $O^q I_q$ cannot vanish unless $ij-2(w-q)-q+1$ becomes negative, which is impossible. For

$$ij-2(w-q)-q+1 = ij-2w+q+1, \text{ and } ij-2w => 0 \text{ by hypothesis.}$$

By taking $O^q I_q$ as an *image*, so to say, of I_q we shall be able to obtain a limit to the number of I_q's by obtaining a limit to the number of their images. In fact, taking the *image* $O^q I_q$ of each of the $D(w-q;\ i,\, j)$ linearly independent invariants of the type $w-q;\ i,\ j$ (this is what is meant by the I_q's) and giving q all possible values from 0 to w inclusive, the total number of these images is obviously

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* The result of operating on I with O and Ω each q times, the two operations following each other according to any law of distribution whatever, will always be a numerical multiple of I; but the value of this multiple will differ for different laws of distribution.

$$D(w;i,j)+D(w-1;i,j)+\dots+D(0;i,j).$$

Each of them will be a gradient of the weight $w-q+q$ (*i. e.* of weight w), and will consist of terms of weight w, degree i, and extent j. The total number of such terms will be the number of ways of making up w with i of the numbers $0, 1, 2, 3, \dots j$, or with the usual notation $(w;\ i,\ j)$. If, then, it can be shown that none of these forms are linearly connected, then, inasmuch as they are all functions of the same $(w;i,j)$ arguments, it will follow that their total number cannot exceed $(w;i,j)$. *I. e.* we shall have shown that

$$D(w;i,j)+D(w-1;i,j)+D(w-2;i,j)+\dots+D(0;i,j)$$

cannot exceed

$$\Delta(w;i,j)+\Delta(w-1;i,j)+\Delta(w-2;i,j)+\dots+\Delta(0;i,j),$$

and by the ballot-box principle, as already stated (inasmuch as no D is less than its corresponding Δ), it will follow that each D is the same as the corresponding Δ, and the theorem to be proved is established.

The proof of this independence is easy. For 1° suppose that there is any linear relation between the forms

$$O^qI_q,\ O^qI'_q,\ O^qI''_q,\ \dots,$$

for each of which the value of q is the same. Denoting these forms by

$$P_q,\ P'_q,\ P''_q,$$

let the relation in question be

$$\lambda P_q+\lambda' P'_q+\lambda'' P''_q+\dots=0.$$

Then
$$\lambda\Omega^qP_q+\lambda'\Omega^qP'_q+\lambda''\Omega^qP''_q+\dots=0.$$

But each argument Ω^qP_q is of the form $\Omega^qO^qI_q$, and since this is equal to I_q multiplied by a number which does not vanish,* we have a linear relation between $I_q,\ I'_q,\ I''_q,\ \dots$, viz.

$$\lambda I_q+\lambda' I'_q+\lambda'' I''_q+\dots=0;$$

i. e. the I_q's would not be linearly independent, contrary to hypothesis. Thus the *images* ($O^qI_q,\ O^qI'_q,\ O^qI''_q\ \dots$) belonging to invariants of the same type $w-q$; i,j cannot be linearly connected.

2°. I say that the images of invariants of different types cannot be linearly connected. For let $q,\ q',\ q'',\ \dots$ arranged in descending order of magnitude,

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* In fact, remembering that the excess of the type $w-q;i,j$ is $ij-2(w-q)=\eta+2q$, we find

$$\Omega^qO^qI_q=1.2.3\dots q\{(\eta+2q)(\eta+2q-1)\dots(\eta+q+1)\}I_q,$$

in which both η and q are *positive* integers.

be the different values of q in the images supposed to be linearly related. The result of operating with Ω^q on any image of the form $O^{q'}I_{q'}$ is to bring it to the form $\Omega^{q-q'}\Omega^{q'}O^{q'}I_{q'}$, which is a multiple of $\Omega^{q-q'}I_{q'}$, and therefore vanishes. But Ω^q, acting on any of the images O^qI_q, $O^qI'_q$, ... , will, as we have seen, bring back the multiple of I_q; thus the operation of Ω^q on the supposed relation will give a linear equation connecting I_q, I'_q, I''_q, ... , and for the same reason as before this is impossible. Hence there can be no linear relation whatever between the images of the invariants whose types extend from $w; i, j$ to $0; i, j$, and the number of these images will accordingly be not greater than $(w; i, j)$, as was to be proved.

It is well worthy of notice that $D(w; i, j)$ may be zero, but obviously cannot be negative, as it denotes a number of things which may have any value from zero upwards. Hence follows a remarkable theorem in the pure theory of partitions which it would be extremely difficult to prove from first principles, viz. that the difference between the two partition numbers

$$(w; i, j) - (w - 1; i, j)$$

can never be negative when $ij - 2w \mathrel{=\!\!>} 0$. It may be zero, but cannot be less than zero. This explains what I said about the hyperbolic paraboloid $ij - 2w = 0$, where i, j, w are treated as co-ordinates of a point in space. We might call the value of $(w; i, j) - (w - 1; i, j)$ the density of any point i, j, w, and the theorem may then be expressed by saying that at points within or upon the hyperbolic paraboloid the density can never be negative; for points outside this surface it can never be positive.

As regards the analogous formula in the Theory of Reciprocants

$$(w; i, j) - (w - 1; i + 1, j),$$

we do not know that any algebraical surface can be constructed which will enable us to discriminate between the cases in which this difference, say $E(w; i, j)$, is positive or negative. Should such a surface exist, its equation must contain w in a higher degree than the first. Supposing that the above formula represents the actual number of reciprocants, it will follow (and this is confirmed by experience) that there can be no reciprocants to a type of negative excess. For

$$\begin{aligned}&(w; i, j) - (w - 1; i + 1, j)\\ &= (w; i, j) - (w - 1; i, j) - [(w - 1; i + 1, j) - (w - 1; i, j,)]\\ &= (w; i, j) - (w - 1; i, j) - (w - i - 2; i + 1, j - 1).\end{aligned}$$

But if $ij - 2w$ is negative, $(w; i, j) - (w - 1; i, j)$ is zero or negative. Hence $(w; i, j) - (w - 1; i + 1, j)$ is non-positive.

For *satisfied* invariants (those ordinarily so called) $w = \frac{ij}{2}$, and the formula for their number becomes $\left(\frac{ij}{2}; i, j\right) - \left(\frac{ij}{2} - 1; i, j\right)$.

As these form a well-defined class apart, it would have seemed very natural to begin with them in endeavoring to establish the theorem, reserving the theory of unsatisfied invariants (sources of covariants) for future consideration. But to all appearance it would have been very difficult, if not impossible, to have succeeded in dealing with them alone.

This is another example of the law in Heuristic that the whole is easier of deglutition than its part.

LECTURE XII.

Before proceeding further with the development of the pure analytical theory of reciprocants, it may be useful to point out some instances of its relations and applications to geometrical questions.

Using $y_1, y_2, y_3, \dots y_n$ to denote the successive derivatives of y with respect to x,[*] let the complete primitive of the differential equation

$$F(x, y, y_1, y_2, \dots y_n) = 0$$

be

$$\phi(x, y, \lambda, \mu, \nu, \dots) = 0.$$

We can in general so determine the n constants $\lambda, \mu, \nu, \dots$ that the curve ϕ may pass through n given points, and if we take these to be consecutive points on the curve

$$\Phi(x, y) = 0,$$

ϕ and Φ will have a contact of the $(n-1)$th order at a given point of Φ. In order that the curves may have a contact of the nth order at a point whose abscissa is x, the ordinates of Φ and ϕ at that point and their 1st, 2d, ... nth derivatives with respect to x must be the same for both curves. But at every point of ϕ its differential equation

$$F(x, y, y_1, y_2, \dots y_n) = 0$$

has to be satisfied, and therefore the $x, y, y_1, y_2, \dots y_n$ of any point on Φ, at which contact of the nth order with ϕ is possible, must also satisfy the same equation.

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[*] In future $y_1, y_2, y_3, \dots y_n$ will always have this meaning, the derivatives of x with respect to y will be denoted by $x_1, x_2, x_3, \dots$, and whenever the letters $t, a, b, c, \dots$ are used they will stand for $y_1, \frac{y_2}{1.2}, \frac{y_3}{1.2.3}, \frac{y_4}{1.2.3.4}, \dots$ respectively.

Now, suppose that for x and y we substitute given functions of them, X and Y; the curves ϕ and Φ become

$$\phi(X,\ Y,\ \lambda,\ \mu,\ \nu,\ \ldots.) = 0 \text{ and } \Phi(X,\ Y) = 0.$$

Contact of the n^{th} order with the transformed ϕ will therefore be possible at any point of the transformed Φ for which

$$F(X,\ Y,\ Y_1,\ Y_2,\ \ldots.\ Y_n) = 0,$$

where $Y_1,\ Y_2,\ Y_3,\ \ldots.\ Y_n$ are the derivatives of Y with respect to X.

But, unless the function F and the substitutions $X = f_1(x,\ y)$, $Y = f_2(x,\ y)$ are so related that the transformed differential equation

$$F(X,\ Y,\ Y_1,\ Y_2,\ \ldots.\ Y_n) = 0$$

is identical with the untransformed one, the property marked by the contact of the transformed curves will not be identical with that marked by the contact of the untransformed ones.

For example, let $F = y_2$; then the relation between $\phi \equiv y + \lambda x + \mu = 0$ (the complete primitive of $y_2 = 0$) and an arbitrary curve Φ is that the constants λ and μ may be so chosen that the line $y + \lambda x + \mu = 0$ may have a contact of the second order at any point of Φ for which $y_2 = 0$; and the property marked is an inflexion on Φ. But if we make the substitution $X = x^2$, $Y = y^2$, so that the differential equation $y_2 = 0$ is transformed into $\left(\frac{d}{dx^2}\right)^2 y^2 = 0$ and its complete primitive into $y^2 + \lambda x^2 + \mu = 0$, it will still be possible so to choose λ and μ that $y^2 + \lambda x^2 + \mu = 0$ may have a contact of the second order at any point of an arbitrary curve for which $\left(\frac{d}{dx^2}\right)^2 y^2 = 0$, but the property marked, instead of being an inflexion, will be a *contact of the second order with a conic having a pair of conjugate diameters coincident with the co-ordinate axes.*

This property remains unaltered when the co-ordinate axes are interchanged, and therefore the differential equation $\left(\frac{d}{dx^2}\right)^2 y^2 = 0$ will be identical with $\left(\frac{d}{dy^2}\right)^2 x^2 = 0$, in which the variables x and y have changed places. The identity of the two differential equations is easily verified, for

$$\left(\frac{d}{dx^2}\right)^2 y^2 = \frac{1}{2x} \cdot \frac{d}{dx}\left(\frac{y}{x} \cdot \frac{dy}{dx}\right) = \frac{1}{2x} \cdot \left\{\frac{y}{x} \cdot \frac{d^2y}{dx^2} + \frac{1}{x}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x^2} \cdot \frac{dy}{dx}\right\}$$

$$= \frac{1}{2x^3}(xyy_2 + xy_1^2 - yy_1);$$

so that the differential equation may be written

$$xyy_2 + xy_1^2 - yy_1 = 0.$$

Interchanging x and y in this, we have

$$yxx_2 + yx_1^2 - xx_1 = 0,$$

in which, if we write $x_1 = \frac{dx}{dy} = \frac{1}{y_1}$, and $x_2 = \frac{d^2x}{dy^2} = -\frac{y_2}{y_1^3}$, it follows immediately that

$$yxx_2 + yx_1^2 - xx_1 = -\frac{1}{y_1^3}(xyy_2 + xy_1^2 - yy_1),$$

and the identity in question is established.

Such a form as the above, which merely acquires an extraneous factor when the variables are interchanged, might be called a reciprocant, if it were not convenient to restrict the use of the word to forms in which the variables x and y do not appear explicitly. With this limitation, the geometrical property indicated by the evanescence of a reciprocant will be independent of the position of the origin, but not in general independent of the directions of the co-ordinate axes. Thus, we may prove that the equation

$$2y_1y_3 - 3y_2^2 = 0$$

indicates the possibility of 4-point contact with a hyperbola whose asymptotes are *parallel to the co-ordinate axes.* To do this it is sufficient to show that its complete primitive is the equation to such a hyperbola.

Writing the equation in the form

$$\frac{y_3}{y_2} = \frac{3}{2} \cdot \frac{y_2}{y_1},$$

we see that its first integral is

$$\log y_2 = \frac{3}{2} \log y_1 + \text{const.};$$

or, when prepared for a second integration,

$$-\frac{1}{2} \cdot y_1^{-\frac{3}{2}} y_2 = \lambda.$$

Hence

$$y_1^{-\frac{1}{2}} = \lambda x + \mu,$$

$$y_1 = (\lambda x + \mu)^{-2},$$

and finally we obtain the complete primitive

$$\lambda(\nu - y) = (\lambda x + \mu)^{-1},$$

which proves the proposition.

With the notation previously explained, in which $y_1 = t$, $y_2 = 2a$, $y_3 = 6b$, the differential equation is $bt - a^2 = 0$. We have therefore proved that at all points of a general curve for which the Schwarzian $(bt - a^2)$ vanishes, 4-point contact with a hyperbola whose asymptotes are parallel to the co-ordinate axes is possible.

We now consider the important case in which the conditioning differential equation remains unchanged when the axes are orthogonally transformed, and is therefore found by equating to zero an orthogonal reciprocant. The simplest example of this class of equations is that which marks the points of maximum or minimum curvature on a curve. Since these points are points of 4-point contact with a circle, the conditioning differential equation will be that of the circle

$$(x+\lambda)^2 + (y+\mu)^2 + \nu = 0.$$

Differentiating this three times in succession, we have

$$x + \lambda + (y+\mu)\,t = 0,$$
$$1 + t^2 + 2a\,(y+\mu) = 0,$$
$$at + b\,(y+\mu) = 0.$$

Eliminating μ from the last two of these equations, y will disappear at the same time, and the condition for points of maximum or minimum curvature is found to be

$$2a^2t - b\,(1+t^2) = 0.$$

In Salmon's Higher Plane Curves (2d edition, p. 357) the "aberrancy of curvature" is given by the formula

$$\tan\delta = y_1 - \frac{(1+y_1^2)\,y_3}{3y_2^2} = t - \frac{(1+t^2)\,b}{2a^2}.$$

The above differential equation is therefore equivalent to $\delta = 0$.

If we differentiate the radius of curvature $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+t^2)^{\frac{3}{2}}}{2a}$, we find

$$\frac{d\rho}{dx} = \frac{6a^2t\,(1+t^2)^{\frac{1}{2}} - 3b\,(1+t^2)^{\frac{3}{2}}}{2a^2} = 3\,(1+t^2)^{\frac{1}{2}}\tan\delta = 3\tan\delta\,.\,\frac{ds}{dx}.$$

Hence it follows that $$\tan\delta = \frac{1}{3}\,.\,\frac{d\rho}{ds}.$$

The conditioning equation for points at which $\frac{d\rho}{ds}$ or $\tan\delta$ is a maximum or minimum is $\frac{d^2\rho}{ds^2} = 0$; or the same condition may be expressed by $\frac{d\tan\delta}{dx} = 0$.

Now, $$\frac{d\tan\delta}{dx} = \frac{d}{dx}\left\{t - \frac{b\,(1+t^2)}{2a^2}\right\} = 2a - \frac{2c\,(1+t^2)}{a^2} - \frac{2abt}{a^2} + \frac{3b^2(1+t^2)}{a^3}$$

is an orthogonal reciprocant, for it can be expressed in terms of legitimate combinations of $1+t^2$, which is an orthogonal reciprocant of even character, with the three orthogonal reciprocants of odd character,

$$a,\ b(1+t^2)-2a^2t,\ c(1+t^2)-5abt+5a^3.$$

In fact, the above expression for $\dfrac{d\tan\delta}{dx}$, when multiplied by a^3 to clear of fractions, becomes,

$$2a^4-2a^2bt+3b^2(1+t^2)-2ac(1+t^2)$$
$$=\frac{3}{1+t^2}\{b(1+t^2)-2a^2t\}^2+\frac{12a^4}{1+t^2}-2a\{c(1+t^2)-5abt+5a^3\},$$

where the right-hand side is a linear function of orthogonal reciprocants of the same (even) character, so that the combination is legitimate.

Quantities such as $\rho,\ \dfrac{d\rho}{ds},\ \dfrac{d^2\rho}{ds^2},\ \ldots\ldots,$ or $\rho,\ \dfrac{d\rho}{d\phi},\ \dfrac{d^2\rho}{d\phi^2},\ \ldots\ldots,$ where $d\phi$ is the angle subtended by the arc ds at the centre of curvature, have values independent of the particular position of the co-ordinate axes (supposed rectangular), and consequently these values, expressed in terms of $t,\ a,\ b,\ c,\ \ldots$ will be absolute orthogonal reciprocants. A differential equation expressing the condition that any one of these quantities vanishes, or that any one of them has a maximum or minimum value, will also be independent of the position of the rectangular axes, and must therefore be expressible in the form of an orthogonal reciprocant equated to zero.

Mr. Hammond remarks that, since the radii of curvature at corresponding points of a curve and its evolute are ρ and $\dfrac{d\rho}{d\phi}$, the radius of curvature of its n^{th} evolute is $\dfrac{d^n\rho}{d\phi^n}$. The radius of curvature of the n^{th} evolute of any n^{th} involute of a circle is constant, and, consequently, the differential equation of an n^{th} involute to a circle is

$$\frac{d^{n+1}\rho}{d\phi^{n+1}}=0.$$

Writing this in the form

$$\left(\frac{1+t^2}{a}\cdot\frac{d}{dx}\right)^{n+1}\cdot\frac{(1+t^2)^{\frac{3}{2}}}{a}=0,$$

to which it is easily reduced, since

$$\frac{d}{d\phi}=\rho\cdot\frac{d}{ds}=\frac{\rho}{(1+t^2)^{\frac{1}{2}}}\cdot\frac{d}{dx}=\frac{(1+t^2)}{2a}\cdot\frac{d}{dx},$$

we see by what precedes that the left-hand member of the differential equation is an orthogonal reciprocant.

As an example of the class of singularities which next presents itself for consideration, let us find the differential condition which holds at points of contact of the fourth order with a common parabola. This condition is expressible by the differential equation whose complete primitive is

$$(y+\varkappa x)^2+2\lambda x+2\mu y+\nu=0.$$

Differentiating three times in succession, we obtain

$$(y+\varkappa x)(t+\varkappa)+\lambda+\mu t=0,$$
$$2a(y+\varkappa x+\mu)+(t+\varkappa)^2=0,$$
$$b(y+\varkappa x+\mu)+a(t+\varkappa)=0.$$

The arbitrary constants ν and λ do not appear in the last two of these equations, from which, if we eliminate μ, the variables x and y disappear at the same time, and we find $$2a^2-b(t+\varkappa)=0.$$

A final differentiation and elimination give

$$10ab-4c(t+\varkappa)=0,$$
$$4ac-5b^2=0.$$

Points of 5-point contact with a parabola are therefore indicated by the evanescence of the pure reciprocant $4ac-5b^2$. And in general the differential equation $R=0$, where R is any pure reciprocant, indicates a property of a curve which may be called a descriptive singularity, since it is totally unaffected by the arbitrary choice of any two lines on the plane for the axes of co-ordinates. For it was proved in Lecture IX of the present course that if i be the degree and μ the characteristic of R, the substitution of $ly+mx+n$ for x and $l'y+m'x+n'$ for y changes R into $(l'm-lm')^i(lt+m)^{-\mu}R$, so that the differential equation $R=0$ and the geometrical property corresponding to it are left unchanged by the substitution.

Six-point contact with a cubical parabola is another example of a descriptive singularity. Its defining differential equation may be written in any of the following forms:

$$45y_2^3y_5^2-450y_2^2y_3y_4y_5+192y_2^2y_4^3+400y_2y_3^3y_5+165y_2y_3^2y_4^2-400y_3^4y_4=0,$$
$$125a^3d^2-750a^2bcd+256a^2c^3+500ab^3d+165ab^2c^2-300b^4c=0,$$
$$5(9y_2^2y_5-45y_2y_3y_4+40y_3^3)^2+64(3y_2y_4-5y_3^2)^3=0,$$
$$125(a^2d-3abc+2b^3)^2+4(4ac-5b^2)^3=0;$$

or, if we make $a^2d-3abc+2b^3=A$ and $ac-\frac{5}{4}b^2=M$, the equation may be put in the form $$\left(\frac{A}{16}\right)^2+\left(\frac{M}{5}\right)^3=0.$$

In the theory of Binary Forms, when the numerical parameter $\varkappa$ in

$$(a^2d - 3abc + 2b^3)^2 + \varkappa(ac - b^2)^3$$

is so chosen that the highest powers of b cancel each other, the form divides by a^2 and gives the Discriminant of the Cubic

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2.$$

In the parallel theory of Reciprocants the form

$$125A^2 + 256M^3$$

is divisible by a (instead of by a^2), giving

$$125a^3d^2 - 750a^2bcd + 500ab^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c,$$

which may be called the Quasi-Discriminant.

A complete discussion of the differential equation

$$A^2 + \varkappa M^3 = 0$$

is reserved for the next ensuing lecture, in the course of which it will appear that the Quasi-Discriminant equated to zero is the differential equation of the cubical parabola.

LECTURE XIII.

We may integrate the general homogeneous equation in reciprocants extending to d, inclusive, as follows:

Calling $\quad ac - \frac{5}{4}b^2 = M$ and $a^2d - 3abc + 2b^3 = A$,

the equation in question will be of the form

$$A^2 + \varkappa M^3 = 0.$$

But if we write $\quad \beta = \Lambda\alpha^\lambda,$

where β, α are general linear functions of the co-ordinates, say

$$y + mx + n,\ y + m'x + n',$$

we may eliminate the five constants m, n, m', n', λ, and the result will evidently be a pure reciprocant extending to d, inclusive, and, being homogeneous and isobaric, can only be of the form

$$A^2 + \varkappa M^3 = 0,$$

so that it remains only to determine $\varkappa$ in terms of λ, or, which is the same thing, λ in terms of $\varkappa$.

The solution $\beta = \Lambda\alpha^{\lambda}$ implies $\alpha = \Lambda^{-\frac{1}{\lambda}}\beta^{\frac{1}{\lambda}}$. Hence the equation between M and A must be of the form

$$\theta\{(\lambda + p)(p\lambda + 1)\}^{i}M^3 + \{(\lambda + q)(q\lambda + 1)\}^{j}A^2 = 0,$$

where θ is a constant, for otherwise there would be more than one general solution to it. It only remains then to determine the values of p, q, θ, i, j, which may be affected by considering the particular solution $y = x^{\lambda}$.

When $\lambda = 2$, M and A both vanish, and if $\lambda = 2 + \varepsilon$, where ε is an infinitesimal, M and A will each be of the same order as ε (that the first power of ε does not vanish in M or A may be easily verified). Hence $2 + q + \varepsilon$ is of the order ε, and therefore $q = -2$ and $j = 1$.

When $\lambda = -1 + \varepsilon$, M remains finite and A is of the order ε. Hence $p = 1$ and $i = 1$. Thus, the equation is

$$\theta(\lambda + 1)^2M^3 + (\lambda - 2)(2\lambda - 1)A^2 = 0.$$

To find θ, let $\lambda = 3$ and $y = x^3$; then

$$a = 3x,\ b = 1,\ c = 0,\ d = 0,\ M = -\frac{5}{4},\ A = 2,$$

so that $$-\theta.\frac{5^3}{4} + 5.4 = 0,\ \theta = \frac{16}{25},$$

and finally $$16(\lambda + 1)^2M^3 + 25(2\lambda^2 - 5\lambda + 2)A^2 = 0$$

has for its integral $$\beta = \Lambda\alpha^{\lambda}.$$

If $\lambda = \infty$, we may make

$$y = \left(1 + \frac{x}{\lambda}\right)^{\lambda l} = e^{lx},$$

and, consequently, $\beta = e^{l\alpha}$, which contains five independent arbitrary constants, will be the general integral.

For a parallel method of deducing the Integral of $A^8 + \varkappa\Delta^3 = 0$, where Δ (our future $AC - B^2$) is the projective reciprocant whose letters go up to f, see Halphen's Thèse sur les Invariants Différentiels, Paris, 1878.

Mr. Hammond has succeeded in deducing the equation between A and M from the primitive $\beta = \Lambda\alpha^{\lambda}$ by direct elimination, as shown in what follows. Possibly he, or some other algebraist, may eventually succeed in the more difficult task of obtaining the Differential Equation to $\gamma = \beta^{\lambda}\alpha^{1-\lambda}$ (*i. e.* the linear relation between A^8 and Δ^3) by some similar direct process.

Differentiating the equation $\beta\alpha^{-\lambda} = \lambda$ three times in succession, and observing that, since $\alpha = y + mx + n$ and $\beta = y + m'x + n'$,

$$\alpha'' = \beta'' = \frac{d^2y}{dx^2} = y_2,$$

we have

$$\alpha\beta' - \lambda\alpha'\beta = 0,$$

$$y_2(\alpha - \lambda\beta) + (1 - \lambda)\alpha'\beta' = 0,$$

$$y_3(\alpha - \lambda\beta) + y_2\{(2 - \lambda)\alpha' + (1 - 2\lambda)\beta'\} = 0.$$

From the last two of these three equations we obtain, by eliminating $(\alpha - \lambda\beta)$,

$$y_3(1 - \lambda)\alpha'\beta' - y_2^2\{(2 - \lambda)\alpha' + (1 - 2\lambda)\beta'\} = 0;$$

or, writing

$$y_2 = 2a,\ y_3 = 6b,\ 2 - \lambda = 3q^2,\ 1 - 2\lambda = -3r^2,\ 1 - \lambda = q^2 - r^2,$$

and dividing by $\alpha'\beta'$, the equation assumes the form

$$\frac{b}{2a^2}(q^2 - r^2) = \frac{q^2}{\beta'} - \frac{r^2}{\alpha'}.$$

Differentiating again, remembering that $\alpha'' = \beta'' = 2a$, and $\frac{da}{dx} = 3b$, $\frac{db}{dx} = 4c$, we find

$$\frac{4ac - 6b^2}{4a^4}(q^2 - r^2) = -\frac{q^2}{\beta'^2} + \frac{r^2}{\alpha'^2}.$$

The elimination of β' between this and the equation immediately preceding it gives

$$\frac{4ac - 6b^2}{4a^4}(q^2 - r^2)q^2 + \left\{\frac{b}{2a^2}(q^2 - r^2) + \frac{r^2}{\alpha'}\right\}^2 - \frac{q^2r^2}{\alpha'^2} = 0.$$

Writing in this $4ac - 5b^2 = 4M$, we obtain by an easy reduction

$$4q^2M\alpha'^2 = r^2\{2a^2 - b\alpha'\}^2,$$

and, taking the square root of each side,

$$\alpha'(2q\sqrt{M} + rb) - 2a^2r = 0.$$

A final differentiation gives

$$\alpha'\left(\frac{qM'}{\sqrt{M}} + 4cr\right) + 2a(2q\sqrt{M} - 5br) = 0.$$

Finally, eliminating α', we obtain

$$(2q\sqrt{M} + rb)(2q\sqrt{M} - 5rb) + ar\left(4cr + \frac{qM'}{\sqrt{M}}\right) = 0.$$

Hence

$$4Mq^2 + qr\left(\frac{aM'}{\sqrt{M}} - 8b\sqrt{M}\right) + r^2(4ac - 5b^2) = 0;$$

or,

$$4(q^2 + r^2)M^{\frac{3}{2}} + qr(aM' - 8bM) = 0.$$

Now,

$$M' = \frac{dM}{dx} = \frac{d}{dx}\left(ac - \frac{5b^2}{4}\right) = 5ad - 7bc,$$

and, consequently,

$$aM' - 8bM = a(5ad - 7bc) - b(8ac - 10b^2) = 5(a^2d - 3abc + 2b^3) = 5A;$$

so that we may write

$$4(q^2 + r^2)M^{\frac{3}{2}} = -qr(aM' - 8bM) = -5qrA;$$

or, $$16(q^2 + r^2)^2M^3 - 25q^2r^2A^2 = 0,$$

where $$3q^2 = 2 - \lambda \text{ and } -3r^2 = 1 - 2\lambda.$$

Replacing q^2 and r^2 by their expressions in terms of λ, the differential equation becomes

$$16(\lambda + 1)^2M^3 + 25(2\lambda^2 - 5\lambda + 2)A^2 = 0.$$

Some special cases may be noticed.

When $\lambda = 2$ or $\frac{1}{2}$, the equation reduces to $M = 0$, which is the differential equation of the common parabola previously obtained.

When $\lambda = 3$ or $\frac{1}{3}$, we obtain $256M^3 + 125A^2 = 0$ for the equation of the cubical parabola, where the expression on the left-hand side is the Quasi-Discriminant.

When $\lambda = -1$, we find $A = 0$ for the differential equation of the general conic.

When λ is an imaginary cube root of negative unity, so that $\lambda^2 - \lambda + 1 = 0$, we have $$(\lambda + 1)^2 + (2\lambda^2 - 5\lambda + 2) = 0,$$

and the differential equation becomes

$$16M^3 - 25A^2 = 0.$$

We shall subsequently avail ourselves of this result in finding the complete primitive of the Halphenian Δ.

In the case where λ is infinite, from the complete primitive $\beta = e^{l\alpha}$ we first eliminate the exponential function and afterwards the arbitrary constant l.

Thus we find $$\beta' = l\alpha'\beta \text{ and } \frac{y_2}{\beta'} = \frac{y_2}{\alpha'} + \frac{\beta'}{\beta};$$

or, $$y_2\beta(\alpha' - \beta') - \alpha'\beta'^2 = 0.$$

Hence $$y_3\beta(\alpha' - \beta') - y_2\beta'(\alpha' + 2\beta') = 0.$$

The elimination of β gives

$$y_3\alpha'\beta' - y_2^2(\alpha' + 2\beta') = 0;$$

or, $$\frac{3b}{2a^2} = \frac{1}{\beta'} + \frac{2}{\alpha'}.$$

Comparing this with the equation previously obtained,

$$\frac{b}{2a^2}(q^2 - r^2) = \frac{q^2}{\beta'} - \frac{r^2}{\alpha'},$$

we see that $q^2 = 1$ and $r^2 = -2$. Substituting these values in the differential equation

$$16(q^2 + r^2)^2 M^3 - 25q^2 r^2 A^2 = 0,$$

it becomes

$$8M^3 + 25A^2 = 0,$$

which is the differential equation corresponding to the complete primitive $\beta = e^{l\alpha}$.

We shall hereafter consider in detail the theory of that special class of pure reciprocants (M. Halphen's Differential Invariants) which retain their form when any homographic substitution is impressed on the variables; *i. e.* when, instead of x and y, we write

$$\frac{lx + my + n}{l''x + m''y + n''} \text{ and } \frac{l'x + m'y + n'}{l''x + m''y + n''}.$$

Since perspective projection is the geometrical equivalent of homographic substitution, it follows from the definition of Differential Invariants that they are connected with the properties and relations of curves which remain unaffected by perspective projection. For this reason Differential Invariants are sometimes called Projective Reciprocants. Two reciprocants with which we are familiar belong to this important class. One of them, y_2 or a, vanishes at points of inflexion on the curve $y = f(x)$; the other,

$$9y_2^2 y_5 - 45y_2 y_3 y_4 + 40y_3^3, \text{ or } a_2 d - 3abc + 2b^3,$$

which, for reasons given below, we shall call the Mongian, vanishes at sextactic points; *i. e.* at points where a conic can be drawn having 6-point contact with the given curve.

To illustrate the distinction between a projective and a merely descriptive singularity, consider for an instant the pure reciprocant $4ac - 5b^2$, which, as we have seen, vanishes at all points of a general curve where 5-point contact with a parabola is possible. Now, 5-point contact with a parabola is a descriptive but not a projective singularity; after projection the parabola becomes a general conic, and 5-point contact with it becomes 5-point contact with a general conic, which is not a singularity at all. But inflexions and sextactic points are indelible by projection, and thus belong to the class of projective singularities.

The differential equation to a conic was originally obtained by Monge in the form

$$9y_2^2 y_5 - 45y_2 y_3 y_4 + 40y_3^3 = 0$$

(see Monge, Sur les Équations différentielles des Courbes du Second Degré. Corresp. sur l'École Polytech. Paris, II, 1809–13, pp. 51–54, and Bulletin de la Soc. Philom., Paris, 1810, pp. 87, 88). At the end of the first chapter of his Differential Equations, Boole mentions this form of equation as due to Monge, but without any reference, and adds the remark: "But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms." The theory of Reciprocants, however, furnishes both a simple interpretation of the Mongian equation and an obvious method of integrating it.

To see that the differential equation of a conic is satisfied at the Sextactic points of a given curve, we have only to remember that at such points the derivatives of y with respect to x, up to the fifth order, inclusive, are the same for the given curve as for a conic.

We proceed to show how the Mongian may be integrated. Writing in the above equation

$$y_2 = 2a,\ y_3 = 2.3b,\ y_4 = 2.3.4c,\ y_5 = 2.3.4.5d,$$

it becomes

$$a^2d - 3abc + 2b^3 = 0,$$

where it can hardly fail to be noticed that the left-hand member of the equation is an ordinary Invariant as well as a Reciprocant. It will be proved hereafter that all Differential Invariants possess this double nature.

Now, if $\mu = 3i + w$, where i is the degree and w the weight of any pure reciprocant R, the ordinary theory of eduction shows that

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \frac{a\dfrac{dR}{dx} - \mu bR}{a^{\frac{\mu}{3}+1}}$$

is another pure reciprocant.

When we consider the letters $a, b, c, \ldots$ in any invariant I to mean $\frac{y_2}{2}, \frac{y_3}{2.3}, \frac{y_4}{2.3.4}, \ldots$ the parallel theory of generation for Invariants gives the corresponding theorem that if $\nu = 3i + 2w$, where i is the degree and w the weight of I,

$$\frac{d}{dx}\left(\frac{I}{a^{\frac{\nu}{3}}}\right) = \frac{a\dfrac{dI}{dx} - \nu bI}{a^{\frac{\nu}{3}+1}}$$

is also an invariant.

A strict proof of this theorem will subsequently be given. For present purposes it is sufficient to notice the easily verified special cases of the two theorems

$$\frac{d}{dx}\left(\frac{4ac-5b^2}{a^{\frac{8}{3}}}\right)=\frac{20\,(a^2d-3abc+2b^3)}{a^{\frac{11}{3}}}$$

and

$$\frac{d}{dx}\left(\frac{ac-b^2}{a^{\frac{10}{3}}}\right)=\frac{5\,(a^2d-3abc+2b^3)}{a^{\frac{13}{3}}}.$$

It follows as an immediate consequence that the equation

$$a^2d-3abc+2b^3=0$$

admits of the two first integrals

$$a^{-\frac{8}{3}}\,(4ac-5b^2)=\text{const.}$$

and

$$a^{-\frac{10}{3}}\,(ac-b^2)=\text{const.}$$

Now,

$$a^{-\frac{8}{3}}\,(4ac-5b^2)=\frac{d}{dx}\,(a^{-\frac{5}{3}}b)=-\frac{1}{2}\,\frac{d^2}{dx^2}\,(a^{-\frac{2}{3}});$$

so that the Mongian equation is equivalent to $\frac{d^3}{dx^3}\,(a^{-\frac{2}{3}})=0$, or to $\frac{d^3}{dx^3}\,(y_2^{-\frac{2}{3}})=0$. We thus obtain an integral of the form

$$y_2^{-\frac{2}{3}}=l+2mx+nx^2,$$

from which the complete primitive may be found by two easy integrations. Thus,

$$y_1+p=\int\frac{dx}{(l+2mx+nx^2)^{\frac{3}{2}}}=\frac{m+nx}{(ln-m^2)(l+2mx+nx^2)^{\frac{1}{2}}}$$

gives

$$y+px+q=\frac{1}{ln-m^2}\,(l+2mx+nx^2)^{\frac{1}{2}},$$

which is the equation of a general conic.

By first interchanging the variables x, y in the Mongian equation (whose form remains unaltered by this interchange, since $a^2d-3abc+2b^3$ is a reciprocant) and then integrating three times with respect to x, we should find another integral of the form

$$x_2^{-\frac{2}{3}}=l'+2m'y+n'y^2.$$

The solution may be completed by two integrations, as in the former method.

Mr. Hammond remarks that $\frac{2\,(ac-b^2)}{a^{\frac{10}{3}}}=\frac{d^2}{dt^2}\,(a^{\frac{2}{3}})$, where $t=y_1$. For, since

$$\frac{d}{dt}=\frac{dx}{dt}\cdot\frac{d}{dx}=\frac{1}{2a}\cdot\frac{d}{dx},$$

we have

$$\frac{d}{dt}\,(a^{\frac{2}{3}})=\frac{1}{2a}\cdot\frac{2}{3}\cdot a^{-\frac{1}{3}}.\,3b=\frac{b}{a^{\frac{4}{3}}},$$

and, consequently,

$$\frac{d^2}{dt^2}\,(a^{\frac{2}{3}})=\frac{1}{2a}\cdot\frac{d}{dx}\,(a^{-\frac{4}{3}}b)=2a^{-\frac{10}{3}}\,(ac-b^2).$$

Hence the integral $a^{-\frac{10}{3}}(ac - b^2) = \text{const.}$ previously obtained for the Mongian is equivalent to $\frac{d^2}{dt^2}(a^{\frac{2}{3}}) = \text{constant}$; *i. e.* to $\frac{d^2}{dy_1^2}(y_2^{\frac{2}{3}}) = \text{const.}$ Thus we have another integral of the form

$$y_2^{\frac{2}{3}} = \lambda + 2\mu y_1 + \nu y_1^2,$$

from which it is also easy to pass to the complete primitive.

I add a few general remarks relating to the subject-matter of this and the preceding lecture. Instead of the cumbrous terms Projective Reciprocants or Differential Invariants, it may be better to use the single word Principiants to denominate that crowning class or order of Reciprocants which remain, to a factor près, unaltered for any homographic substitutions impressed on the variables. This is the *species princeps.* If we go back to the *species infima,* we see the beginning of life in the subject. In general Reciprocants, all that is affirmed is that there exist forms-functions of the derivatives of y in regard to x which (to a factor près) remain unaltered when the variables x and y are interchanged, so that $f(y_1, y_2, y_3, \ldots.)$ becomes $\phi(x_1, x_2, x_3, \ldots.)$. The function ϕ only differs from f by the acquisition of an extraneous factor $(-)^\kappa y_1^\mu$; *i. e.*

$$f(y_1, y_2, y_3, \ldots.) = (-)^\kappa y_1^\mu f(x_1, x_2, x_3, \ldots.).$$

A particular species of these general (mixed) reciprocants arises when $f(y_1, y_2, y_3, \ldots.)$, differentiated in regard to y_1, gives a reciprocant. These are Orthogonal Reciprocants, and in them we see the first dawn of free continuous motion as distinguished from mere displacement (or mere interchange of axes). Orthogonal Reciprocants, when x, y are rectangular co-ordinates, remain unaltered (save as to a factor) when the orthogonal axes are moved continuously. A quarter of a revolution of course will reverse their original positions, so that we see the condition of mutual displacement is fulfilled. Thirdly, Reciprocants into whose form the first derivative y_1 does not enter are called Pure. Their form is invariable when the axes (now taken generally) undergo separate displacement (instead of turning round together) in a plane. Here there is a further development, so to say, of life in the subject.

Finally, in Principiants, a particular species of Pure Reciprocants, the invariance remains good, not merely for any position of the axes of reference, but for any homographic deformation of the plane in which they lie, so that the evanescence of a Principiant corresponds to some property of a curve not only intrinsic but indelible by projection, as *ex. gr.* an inflexion, or a double point, or a sextactic point, and so on.

It is clear from this review that the Theory as we have given it goes to the root of the subject, and that the word Reciprocant is rightly chosen as conveying the notion of a property which is common to the entire continuous series of forms bearing that name. All the links of this connected chain are thus comprehended under the general name of Reciprocants.

LECTURE XIV.

The remaining lectures of the course will be devoted to the theory of Pure and Projective Reciprocants. I shall first treat of the existence and properties of the Protomorphs of Invariants and Reciprocants, using the latter system of protomorphs to obtain all the fundamental forms of Reciprocants in the letters a, b, c, d, e. I shall then pass on to the theory of Projective Reciprocants, or Principiants, with its applications contained in M. Halphen's Thèse pour obtenir le grade de docteur ès sciences (Paris, Gauthier Villars, 1878). It will be seen that M. Halphen's very ingenious methods become greatly simplified when his results are read by the light of an important discovery in the theory of Principiants recently made by myself and Mr. Hammond working conjointly, arising out of a theorem put forward by one of my hearers. This theorem, on examination, we found was necessarily erroneous and would fail at the very first step of its application. But although the proposition stated was wrong, it contained an Idea which survives and may be incorporated in a valid and extremely important theorem, which I will endeavor to explain.

A Principiant, besides being an Invariant in the original letters $a, b, c, d, \ldots.$ is also an Invariant in the letters $a, A, B, C, D, \ldots.$ where each capital letter is itself a Reciprocant; and, conversely, every invariant in the capital letters $A, B, C, D, \ldots.$ is a Principiant. The invariants in the capital letters form a system of protomorphs for Principiants, so that every Principiant is either some such invariant simply, or a rational integral function of such invariants divided by some power of a. Thus, for example, it will be proved that the Cubic Criterium (*i. e.* the Principiant which gives, when equated to zero, the differential equation of a cubic curve) may be expressed as the quotient of

$$\frac{9}{64}A^5+\frac{5}{4}A(A^2D-3ABC+2B^3)-(ACE-AD^2-B^2E+2BCD-C^3)$$

by the fifth power of a.

The proof of this theorem is based upon the fact that we can form a series of terms beginning with the Mongian (viz. $a^2d-3abc+2b^3$), say A, B, C, D, ... such that

$$\Omega A=0,$$
$$\Omega B=A\times\frac{a}{2},$$
$$\Omega C=2B\times\frac{a}{2},$$
$$\Omega D=3C\times\frac{a}{2},$$
$$\cdots\cdots\cdots\cdots$$

where

$$\Omega=a\partial_b+2b\partial_c+3c\partial_d+\ldots\ldots,$$

coupled with the fact that every Principiant must be a function of the letters in such series and the small a.

Each consequent of the series A, B, C, D, ... is, so to say, an Invariant relative to its antecedent; it becomes an actual Invariant when its antecedent vanishes.

In the theorem as originally proposed, each letter of the series was derived by the operation of an eductive generator upon the one which precedes. In the true theorem the scale of relation is between three and not two consecutive terms. Calling the letters u_0, u_1, u_2, ... u_i, we have

$$(i+7)u_{i+2}-Gu_{i+1}+(i+1)Mu_i=0,$$

where G is the ordinary eductive generator,

$$4(ac-b^2)\partial_b+5(ad-bc)\partial_c+6(ae-bd)\partial_d+\ldots\ldots,$$

M is the first pure reciprocant after the monomial a, viz. $M=ac-\frac{5}{4}b^2$, $u_0=A=a^2d-3abc+2b^3$, and $6u_1=GA$.

But although, as I have said, the theorem in the form proposed was absolutely erroneous, its proposer has rendered an invaluable service to the theory by the mere suggestion of what turns out to be true, viz. that every Principiant is an Invariant in regard to a known series of Reciprocants considered as simple elements.

To this theorem there is a correlative one, for it will be shown that there exists a series of invariants A_0, A_1, A_2, ..., the first term of which, A_0, is the same as the Mongian A, each of the other terms of the series being a Reciprocant relative to the one that precedes it. In fact, we have

$$\begin{aligned} VA_0 &= 0, \\ VA_1 &= -a^2A_0, \\ VA_2 &= -2a^2A_1, \\ &\dots\dots\dots\dots \\ VA_n &= -na^2A_{n-1}, \end{aligned}$$

where

$$V = 4\left(\frac{a^2}{2}\right)\partial_b + 5ab\partial_c + 6\left(ac + \frac{b^2}{2}\right)\partial_d + \dots,$$

and, as a consequence, every Principiant will be an Invariant in respect to these Invariants and the first small letter a.

Thus, speaking symbolically, we have not only

$$P = R + I$$

(a logical equation meaning that P has the same qualities as both R and I, or that a Principiant is both a Reciprocant and an Invariant), but also

$$P = IR \text{ and } P = II,$$

meaning that a Principiant is an Invariant of Reciprocantive elements, and an Invariant whose elements are themselves Invariants.

I may add that the invariantive elements A_0, A_1, A_2, A_3, ... are defined by the equations

$$\begin{aligned} A_0 &= A, \\ A_1 &= B - \frac{b}{2}A, \\ A_2 &= C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2A, \\ A_3 &= D - 3\left(\frac{b}{2}\right)C + 3\left(\frac{b}{2}\right)^2B - \left(\frac{b}{2}\right)^3A, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

so that any invariant in the reciprocantive elements A, B, C, D, ... is equal to the corresponding invariant in A_0, A_1, A_2, A_3, ... Thus,

$$\begin{aligned} A &= A_0, \\ AC - B^2 &= A_0A_2 - A_1^2, \\ A^2D - 3ABC + 2B^3 &= A_0^2A_3 - 3A_0A_1A_2 + 2A_1^3, \\ AE - 4BD + 3C^2 &= A_0A_4 - 4A_1A_3 + 3A_2^2, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

M. Halphen appears not to have noticed the Principiant $AE - 4BD + 3C^2$, which presents itself naturally when the theory is viewed from our present ground of vantage, but A, $AC - B_2$ and $A^2D - 3ABC + 2B^3$ occur in his Thèse in connection with the curve

$$\alpha = \beta^\lambda \gamma^{1-\lambda},$$

in which α, β, γ are any linear functions of x, y, 1.

When $\lambda = -1$ the differential equation of this curve (the conic $\alpha\beta = \gamma^2$) is $A = 0$, but it is

$$AC - B^2 = 0$$

when λ is a cube root of negative unity, and

$$A^2D - 3ABC + 2B^3 = 0$$

when λ has an arbitrary value.

Before making out an exhaustive table of all the irreducible forms of pure reciprocants in the letters a, b, c, d, e similar to, but not identical with, the corresponding table for invariants, it seems to me desirable to say something of Protomorphs in general; and this will be better understood if we devote a short space to the protomorphs of Invariants. The simplest forms of these are the following well-known ones of alternately the second and third degrees:

$$\begin{aligned}
P_2 &= ac - b^2,\\
P_3 &= a^2d - 3abc + 2b^3,\\
P_4 &= ae - 4bd + 3c^2,\\
P_5 &= a^2f - 5abe + 2acd + 8b^2d - 6bc^2,\\
P_6 &= ag - 6bf + 15ce - 10d^2,\\
P_7 &= a^2h - 7abg + 9acf - 5ade + 12b^2f - 30bce + 20bd^2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The quadratic Protomorphs $P_2, P_4, P_6, \ldots\ldots$, are absolutely unique, for the number of invariants of the type j; $2, j$ is $(j; 2, j) - (j-1; 2, j) = 1$ if j is even, and $= 0$ if j is odd. Their form is so well known that there is no need to dilate upon it here.

The cubic ones $P_3, P_5, P_7, \ldots\ldots$, may be derived from the quadratic ones by means of Cayley's generators, given early in the course, viz.:

$$P = (ac - b^2)\,\partial_b + (ad - bc)\,\partial_c + (ae - bd)\,\partial_d + \ldots\ldots,$$

$$Q = (ac - 2b^2)\,\partial_b + 2\,(ad - 2bc)\,\partial_c + 3\,(ae - 2bd)\,\partial_d + \ldots\ldots$$

Let us first use the P generator

$$P(ac - b^2) = a\,(ad - bc) - 2b\,(ac - b^2) = a^2d - 3abc + 2b^3,$$

$$\begin{aligned}
P\,(ae - 4bd + 3c^2) &= a\,(af - be) - 4b\,(ae - bd) + 6c\,(ad - bc) - 4d\,(ac - b^2)\\
&= a^2f - 5abe + 2acd + 8b^2d - 6bc^2.
\end{aligned}$$

Similarly, we find

$$P(ag - 6bf + 15ce - 10d^2) = a^2h - 7abg + 9acf - 5ade + 12b^2f - 30bce + 20bd^2,$$

and so on.

Let I be any invariant whatever of the type $w;\ i, j$ (satisfied or unsatisfied); then using the original forms of the generators P and Q as given by Cayley (see Lect. IV), we have

$$PI = a(b\partial_a + c\partial_b + d\partial_c + \ldots .)I - ibI,$$
$$QI = a(c\partial_b + 2d\partial_c + 3e\partial_d + \ldots .)I - 2wbI,$$

and, consequently,

$$(jP - Q)I = a\{jb\partial_a + (j-1)cd_b + (j-2)d\partial_c + \ldots .\}I - (ij - 2w)I.$$

If in this formula we write

$$O = jb\partial_a + (j-1)c\partial_b + (j-2)d\partial_c + \ldots . ,$$

it becomes $$(jP - Q)I = aOI - (ij - 2w)bI,$$

which, when I is a satisfied invariant, so that $ij - 2w = 0$ and $OI = 0$, reduces to $$(jP - Q)I = 0,$$

showing that the forms obtained by operating with either P or Q on any satisfied invariant are the same to a numerical factor près.

Now, each quadratic protomorph is a satisfied invariant (for when $w = j$ and $i = 2$, $ij - 2w = 0$), and therefore the cubic protomorphs found by operating on the quadratic ones with Q will only differ by a numerical factor from those already obtained by the operation of P. But we must not conclude from this that the cubic protomorphs are unique. Their number is in fact given by the formula $$(j;\ 3, j) - (j-1;\ 3, j),$$

where it is obvious that

$$(j-1;\ 3, j) = (j-1;\ 3, j-1);$$

so that the above formula may be written

$$(j;\ 3, j) - (j-1;\ 3, j-1),\ \text{ or say } \Delta(j;\ 3, j).$$

Now, there is a simple rule for finding $(j;\ 3, j)$; it is the nearest integer to $\frac{(j+3)^2}{12}$. From the following table, obtained by the use of this rule,

$j =$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$(j;\ 3, j) =$	2	3	4	5	7	8	10	12	14	16	19	21	24	27
$\Delta(j;\ 3, j) =$		1		1		1		2		2		2		3

it may be seen that for any odd number $j \mathrel{=}\!> 9$ there are two or more forms of extent j equally entitled to rank as protomorphs. If l be the last letter which occurs in one of these forms, its first term will of course be a^2l; the difference between any two such forms will not involve the letter l, and will only extend to k, but will still be of the same (potential) extent as l.

The property of the protomorphs $a, P_2, P_3, P_4, \ldots$ is that every invariant is a rational integral function of them divided by some power of a, as appears from the fact that Q, any given rational integral function whatever of the letters $a, b, c, d, e, \ldots$, may obviously be expressed as a rational integral function of $a, b, P_2, P_3, P_4, \ldots$ divided by some power of a. Thus,

$$Q = a^{-m}\phi(a, b, P_2, P_3, P_4, \ldots).$$

Suppose Q to be an invariant I; then

$$Ia^m = \phi(a, b, P_2, P_3, P_4, \ldots),$$

and, consequently,

$$\Omega(Ia^m) = \frac{d\phi}{da}\Omega a + \frac{d\phi}{db}\Omega b + \frac{d\phi}{dP_2}\Omega P_2 + \frac{d\phi}{dP_3}\Omega P_3 + \ldots,$$

where Ω is the annihilator for invariants; so that

$$\Omega(Ia^m) = 0,\ \Omega a = 0,\ \Omega P_2 = 0,\ \Omega P_3 = 0, \ldots$$

We have therefore

$$\frac{d\phi}{db}\Omega b = a\frac{d\phi}{db} = 0.$$

Hence ϕ does not contain b, but is a rational integral function of the protomorphs alone, and

$$I = a^{-m}\phi(a, P_2, P_3, P_4, \ldots).$$

I shall show how to obtain a similar scale of forms possessing like properties for pure reciprocants.

LECTURE XV.

A Protomorph may be defined as a form whose weight is equal to its actual extent, so that its type is $j; i, j$. The first protomorph is a, which corresponds to $j = 0$. For higher values of j it follows immediately from the definition that every protomorph will contain a term $a^{i-1}l$, in which the letter of highest extent appears only in the first degree multiplied by a power of the first letter. The existence of this term enables us to instantly recognize a protomorph. As in the case of invariants, it will be shown that every pure reciprocant is either a rational integral function of protomorphs or else such a function divided by some power of a. But first it will be better to prove *à priori* their existence and exhibit examples of them for the earlier values of j.

It was proved, in Lecture IX, that the number of pure reciprocants of the type $w; i, j$ is at least equal to

$$(w; i, j) - (w - 1; i + 1, j).$$

Now, obviously, the number of partitions of w into i parts not exceeding $w + \varepsilon$ is the same as the number of partitions of w into i parts not exceeding w, so that

$$(w; i, w + \varepsilon) = (w; i, w);$$

and since, by a well-known theorem, $(w; i, j) = (w; j, i)$, we see that

$$(w; w + \varepsilon, j) = (w; j, w + \varepsilon) = (w; j, w) = (w; w, j),$$

a result which follows more immediately from the consideration that the partitions of $w; w + \varepsilon, j$ differ only from those of $w; w, j$ by ε columns of zeros, as we see in the annexed example:

3; 5, 3	3; 3, 3
30000	300
21000	210
11100	111

Hence, if $w = j$, and $i => j$, we have

$$(w; i, j) = (j; j, j)$$

and

$$(w - 1; i + 1, j) = (j - 1; j - 1, j - 1).$$

Thus, the number of pure reciprocants of the type $j; j, j$ is

$$(j; j, j) - (j - 1; j - 1, j - 1),$$

in other words, the difference between the indefinite partitions of j and those of $j - 1$. Expressed by means of generating functions, this difference is the coefficient of x^j in

$$\frac{1 - x}{(1 - x)(1 - x^2)(1 - x^3) \ldots . (1 - x^j)}$$

$=$ coefficient of x^j in the expansion of

$$\frac{1}{(1 - x^2)(1 - x^3) \ldots . (1 - x^j)}.$$

This coefficient is a positive integer for all values of j (except $j = 1$, when it is zero), which proves the existence of reciprocants of the type $j; j, j$ when j has any value except unity.

But we wish to prove the existence of one or more reciprocants of the type $j;\ j, j$ which actually contain a term of the form $a^{j-1}l$, where the letter l is of extent j. The number of such forms is the difference between the number of pure reciprocants of the types $j; j, j$ and $j; j, j-1$.

Now, the number of linearly independent pure reciprocants of the type $j; j, j$ has just been shown to be

$$(j;j,j)-(j-1;j-1,j-1).$$

And, in like manner, that of the linearly independent reciprocants of the type $j;j,j-1$ is

$$\begin{gathered}(j;\ j,j-1)-(j-1;\ j+1,j-1)\\=(j;\ j,j-1)-(j-1;\ j-1,j-1).\end{gathered}$$

The difference between these two numbers is therefore

$$(j;\ j,j)-(j;\ j,j-1)=1.$$

For the only partition not common to the two types is $j.0^{j-1}$, made up of one j and $j-1$ zeros, which belongs to the first type, but not to the second. Hence reciprocants of the type $j; j, j$ contain one term which those of the type $j; j,. j-1$ do not, and which can only be $a^{j-1}l$. This proves the existence of protomorphs.

In the latter part of the above proof we have assumed the truth of the theorem, which, however probable, is not demonstrated, that the number of reciprocants of the type $w;\ i, j$ is $(w;\ i, j)-(w-1;\ i+1, j)$ and *no more* [that concerns the subtrahend, viz.: $(j;\ j, j-1)-(j-1;\ j-1, j-1)$].

We shall, however, have an independent method of arriving at Protomorphs by direct generation, just as we saw that all the cubic protomorphs to invariants were derivable by direct operation of generators from the quadratic ones.

The difference between the two cases is that the lowest degree of Invariantive Protomorphs fluctuates alternately between 2 and 3. For Reciprocantive Protomorphs the lowest degree corresponding to a given extent fluctuates, but has a tendency to rise, and goes on progressing until it exceeds any assignable number.

It is interesting to find what the degrees are for successive values of j. The calculations required are greatly facilitated by an extensive table of partitions given by Euler in 1750, and partly reproduced by Cayley in the *American Journal of Mathematics*, Vol. IV, Part 3. In the table as presented by Cayley, the number in column j and line i means the number of ways of partitioning j into exactly i parts (zeros excluded). Hence, to find the number of ways of partitioning j into i parts or fewer; *i. e.* to find $(j; i, \infty)$ or its equivalent $(j; i, j)$, we must add up the numbers in the 1st, 2d, 3d, ... ith lines of column j.

When these summations are made we obtain the subjoined table:

EXTENT $j =$

DEGREE $i =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10
3	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37
4	1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64	72	84
5	1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84	101	119	141
6	1	1	2	3	5	7	11	14	20	26	35	44	58	71	90	110	136	163	199
7	1	1	2	3	5	7	11	15	21	28	38	49	65	82	105	131	164	201	248
8	1	1	2	3	5	7	11	15	22	29	40	52	70	89	116	146	186	230	288

The number of pure reciprocants of the type $j; i, j$ is

$$(j;\ i, j) - (j - 1;\ i + 1, j) = (j;\ i, j) - (j - 1;\ i + 1, j - 1).$$

To find the minimum degree for protomorphs of extent j we have therefore only to see for what value of i any figure in the j column first becomes greater than the figure in the column to the left one place lower down. The fluctuations of the minimum degree are indicated by the dark irregularly waving line which runs through the table.

Accordingly, we find that the types of the protomorphs, omitting w, which is always equal to j, are as follows:

(2, 2), (3, 3), (3, 4), (4, 5), (3, 6), (4, 7), (4, 8), (5, 9), (5, 10), (5, 11), (5, 12), ... ,

whereas for invariants they are

(2, 2), (3, 3), (2, 4), (3, 5), (2, 6), (3, 7), (2, 8), (3, 9), (2, 10), (3, 11), (2, 12), ...

Corresponding to the extents

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ... ,

the lowest degrees of the Reciprocantive Protomorphs are

2, 3, 3, 4, 3, 4, 4, 5, 5, 5, 5, ...

Contrast this with the regularly fluctuating series

2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, ... , which shows the minimum degrees of invariantive protomorphs for successive extents.

It may be proved, from known formulae in the theory of partitions, that as the extent increases the minimum degree of reciprocantive protomorphs increases (on the whole) and ultimately becomes infinite when the extent is so.

The apparent number of protomorphs to the several types is

$$\begin{array}{cccccccccccc} (2,2), & (3,3), & (3,4), & (4,5), & (3,6), & (4,7), & (4,8), & (5,9), & (5,10), & (5,11), & (5,12), & \ldots \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 2 & 3 & \end{array}$$

The explanation of this multiplicity is the same as that previously given for the case of invariants: the difference between any two protomorphs of a given type $j;\, i, j$ will be a reciprocant (no longer a protomorph) of the type $j;\ i, j-1$.

For the only term containing the letter l (of extent j) will disappear from the result of subtraction; and, accordingly, the above numbers, each diminished by unity, will give the numbers of a set of reciprocants of the same degree-weight as the protomorphs, but of a smaller (actual) extent.

Assuming that the number of pure reciprocants of the type $w;\, i, j$ is correctly given by the formula

$$(w;\, i, j) - (w-1;\ i+1, j),$$

Euler's great table of partitions, already referred to, enables us to carry on the determination of the minimum degree and multiplicity of protomorphs for all extents as far as 59.

If m is the multiplicity corresponding to the minimum degree i of a reciprocantive protomorph whose extent is j, we form without difficulty, using only the principles explained above, the following table:

$j=$	0	1	2	3	4	5	6	7	8	9	10	11	
$i=$	1	–	2	3	3	4	3	4	4	5	5	5	,
$m=$	1	0	1	1	1	1	1	1	2	3	4	2	

$j=$	12	13	14	15	16	17	18	19	20	21	22	23	
$i=$	5	6	6	6	6	7	7	7	7	7	8	8	,
$m=$	3	6	8	5	5	15	18	12	12	2	40	32	

$j=$	24	25	26	27	28	29	30	31	32	33	34	35	
$i=$	8	8	8	9	9	9	9	10	10	10	10	10	,
$m=$	32	14	6	84	82	58	45	207	211	180	161	102	

$j=$	36	37	38	39	40	41	42	43	44	45	46	47
$i=$	10	11	11	11	11	11	11	12	12	12	12	12
$m=$	45	482	469	391	320	167	13	1126	1064	881	687	337

$j=$	48	49	50	51	52	53	54	55	56	57	58	59
$i=$	13	13	13	13	13	13	13	14	14	14	14	14
$m=$	2829	2666	2492	2097	1643	892	26	6394	6017	5227	4266	2755

Notice the repetitions of i indicated by the series

$$1^1,\ 0^1,\ 2^1,\ 3^2,\ 4^1,\ 3^1,\ 4^2,\ 5^4,\ 6^4,\ 7^5,\ 8^5,\ 9^4,\ 10^6,\ 11^6,\ 12^5,\ 13^7,\ 14^{5+?}.$$

It will be observed that there is a general tendency of the number of equal values of i to increase, but that this is subject to occasional fluctuations. When $j=5$, $i=4$; but when $j=6$, $i=3$, so that the minimum value of i recedes. After this point is reached, i either advances or remains stationary, but never recedes.

In order actually to find the protomorphs, we may use the annihilator V. This was my original method of obtaining them; a shorter way, analogous to that used by Halphen for differential invariants (principiants), has been previously mentioned, but it will be instructive to begin with the method of indeterminate coefficients. In the first place we have the form a of weight 0, which is annihilated by

$$V = 2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\,\partial_d + (7ad + 7bc)\,\partial_e + \ldots.$$

For weight 1 there is no pure reciprocant. We could not make $R=\lambda a^{i-1}b$, for then $VR = 2\lambda a^{i+1}$, which cannot vanish unless $\lambda = 0$ and consequently $R=0$.

To find the Protomorph of extent 2, assume $R = \lambda ac + \mu b^2$; then

$$VR = 4\mu a^2 b + 5\lambda a^2 b = (4\mu + 5\lambda)\,a^2 b.$$

Hence λ and μ are proportional to 4 and -5, and we may write

$$R = 4ac - 5b^2.$$

For extent 3, assuming $R = \lambda a^2 d + \mu abc + \nu b^3$, we have

$$VR = 2\mu a^3 c + 6\nu a^2 b^2 + 5\mu a^2 b^2 + 6\lambda a^3 c + 3\lambda a^2 b^2,$$

which vanishes when

$$2\mu + 6\lambda = 0,\quad 6\nu + 5\mu + 3\lambda = 0.$$

We may therefore write $\lambda = 1$, $\mu = -3$, $\nu = 2$, and thus obtain

$$R = a^2 d - 3abc + 2b^3.$$

For extent 4 the table of minimum degrees indicates the existence of a protomorph of degree 3. To find its value we assume

$$R = \kappa a^2e + \lambda abd + \mu ac^2 + \nu b^2c.$$

Operating with V, we find

$$\begin{array}{rccc} & a^3d & a^2bc & ab^3 \\ VR = & 2\lambda & 4\nu & . \\ & . & 10\mu & 5\nu \\ & . & 6\lambda & 3\lambda \\ & 7\kappa & 7\kappa & . \end{array}$$

In order that VR may vanish, we must have

$$2\lambda + 7\kappa = 0, \quad 4\nu + 10\mu + 6\lambda + 7\kappa = 0, \quad \text{and} \quad 5\nu + 3\lambda = 0.$$

To avoid fractions, let $\kappa = 50$; then $\lambda = -175$, $\nu = 105$, and $\mu = 28$; thus,

$$R = 50a^2e - 175abd + 28ac^2 + 105b^2c;$$

whereas, the protomorph of extent 4 for Invariants is $ae - 4bd + 3c^2$. There is no reciprocant of degree 2 weight 4 to correspond to this.

LECTURE XVI.

By using the generator for pure reciprocants instead of the annihilator V, we readily obtain the protomorph of extent 5 and of the fourth degree whose existence is indicated in the previously given table of minimum degrees. We have only to operate on the protomorph of degree 3 and extent 4 with

$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + 7(af - be)\partial_e + \ldots.$$

Thus,

$$\begin{aligned} & G(50a^2e - 175abd + 28ac^2 + 105b^2c) \\ &= 4(ac - b^2)(-175ad + 210bc) \\ &+ 5(ad - bc)(56ac + 105b^2) \\ &+ 6(ae - bd)(-175ab) \\ &+ 7(af - be)(50a^2). \end{aligned}$$

Rejecting the numerical factor 35, which is common to all the terms in the result, and at the same time writing the terms themselves in reverse order, we have

$$\begin{aligned} & 10a^2(af - be) - 30ab(ae - bd) + (ad - bc)(8ac + 15b^2) + 4(ac - b^2)(-5ad + 6bc) \\ &= 10a^3f - 40a^2be - 12a^2cd + 65ab^2d + 16abc^2 - 39b^3c, \end{aligned}$$

which is the protomorph in question.

The form just found is irreducible, as indeed it ought to be, since the minimum degree for extent 5 is greater than that for extent 4 by unity, which exactly corresponds with the unit increase of degree due to the operation of G. But if we use G to generate a protomorph of extent 4 from that of extent 3, the resulting form will be reducible. In fact,

$$\begin{aligned}&G\,(a^2d - 3abc + 2b^3)\\ &= 4\,(ac - b^2)(-3ac + 6b^2) + 5\,(ad - bc)(-3ab) + 6\,(ae - bd)\,a^2\\ &= 3\,(2a^3e - 7a^2bd - 4a^2c^2 + 17ab^2c - 8b^4).\end{aligned}$$

If now we write

$$\begin{aligned}&ac - \frac{5}{4}\,b^2 = M,\\ &a^2d - 3abc + 2b^3 = A,\\ &a^3e - \frac{7}{2}\,a^2bd - 2a^2c^2 + \frac{17}{2}\,ab^2c - 4b^4 = B,\end{aligned}$$

we have shown that

$$GA = 6B\,.$$

But

$$\begin{aligned}50B + 128M^2 &= 25\,(2a^3e - 7a^2bd - 4a^2c^2 + 17ab^2c - 8b^4) + 8\,(4ac - 5b^2)^2\\ &= a\,(50a^2e - 175abd + 28ac^2 + 105b^2c);\end{aligned}$$

so that B is reducible, being expressible as a rational integral function of a, M, and the previously obtained protomorph of degree 3 and extent 4.

The general theory of the generator G is contained in that of the differentiation of absolute reciprocants, in which, if $\mu = 3i + w$, where w is the weight and i the degree of any pure reciprocant R, we have

$$\frac{R}{a^{\frac{\mu}{3}}} = \pm\,\frac{R_1}{a_1^{\frac{\mu}{3}}}\,,$$

and, consequently,

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \pm\,\frac{dy}{dx}\cdot\frac{d}{dy}\left(\frac{R_1}{a_1^{\frac{\mu}{3}}}\right),$$

where R_1 and a_1 are what R and a become when x and y are interchanged. Hence

$$\frac{a\,\dfrac{dR}{dx} - \dfrac{\mu}{3}\,R\,\dfrac{da}{dx}}{a^{\frac{\mu}{3}+1}}\,,$$

and therefore also the numerator of this fraction is a reciprocant.

Remembering that

$$\frac{da}{dx}=3b,\ \frac{db}{dx}=4c,\ \frac{dc}{dx}=5d,\ \ldots\ldots,$$

the numerator may be written

$$a\frac{dR}{dx}-\mu bR=GR.$$

The ordinary expression for G is found by writing

$$a\frac{d}{dx}-\mu b=a\left(3b\partial_a+4c\partial_b+5d\partial_c+\ldots.\right)$$
$$-b\left(3a\partial_a+4b\partial_b+5c\partial_c+\ldots.\right).$$

If the actual extent of R is j, that of GR is $j+1$; for the operation of G introduces an additional letter. Both the weight and degree are also increased by unity. Thus, the type of R being $w;\ i,j$, that of GR is $w+1;\ i+1,j+1$. Suppose the weight of R to be equal to its actual extent; then R is a protomorph of the type $j;\ i,j$, and GR, whose type is $j+1;\ i+1,\ j+1$, is also a protomorph. This proves the existence of protomorphs for every possible extent. Starting with the form $4ac-5b^2$ we obtain, by successive eduction, a series of protomorphs of the type $j;j,j$ for which the general expression is

$$G^{j-2}(4ac-5b^2),$$

where j has any of the values $2,\ 3,\ 4,\ \ldots.$

If R is a protomorph of minimum degree, GR (if irreducible) will also be a protomorph of minimum degree. Hence the minimum degree can never increase by more than one unit when the extent is increased by unity.

The second educt G^2R is always reducible; for

$$G^2R=\left\{a\frac{d}{dx}-(\mu+4)b\right\}\left(a\frac{d}{dx}-\mu b\right)R$$
$$=\left\{a^2\frac{d^2}{dx^2}-(2\mu+1)ab\frac{d}{dx}-4\mu ac+\mu(\mu+4)b^2\right\}R.$$

Combining this with $M=ac-\frac{5}{4}b^2$, we have

$$5G^2R+4\mu(\mu+4)MR=a\left\{5a\frac{d^2}{dx^2}-5(2\mu+1)b\frac{d}{dx}+4\mu(\mu-1)c\right\}R,$$

where the right-hand side is divisible by a, showing that the degree of G^2R is always depressible by unity. R being a protomorph of degree i and extent j,

$$\left\{5a\frac{d^2}{dx^2}-5(2\mu+1)\frac{d}{dx}+4\mu(\mu-1)c\right\}R$$

is one of degree $i+1$ and extent $j+2$. Hence we may conclude that an increase in the minimum degree for protomorphs cannot be immediately followed by another increase; for, if this were possible, the minimum degree for extent $j+2$ would be $i+2$, instead of being $i+1$ at most.

This conclusion is in accordance with the sequence of the values of i in the table of minimum degrees, and as far as it goes confirms the exactitude of the formula $(w; i, j) - (w-1; i+1, j)$ for the number of pure reciprocants which was assumed in calculating the table.

The method previously employed to prove that every invariant is a rational integral function of protomorphs, or such function divided by a power of a, may be very easily extended to the case of reciprocants.

In the first place, it is obvious that every rational integral function of the letters $a, b, c, d, \ldots$ is by successive substitutions reducible to the form

$$a^{-\theta}\Phi(a, b, P_2, P_3, P_4, \ldots . P_j),$$

where P_j means the protomorph of extent j.

Let any reciprocant R be put under this form; then

$$a^{\theta}R = \Phi(a, b, P_2, P_3, P_4, \ldots . P_j),$$

and, consequently,

$$V(a^{\theta}R) = \frac{d\Phi}{da}Va + \frac{d\Phi}{db}Vb + \frac{d\Phi}{dP_2}VP_2 + \ldots . + \frac{d\Phi}{dP_j}VP_j.$$

Now, V annihilates R, a, P_2, P_3, ... P_j, since these are all pure reciprocants. Hence the above identity reduces to $\frac{d\Phi}{db}Vb = 0$, from which (since Vb does not vanish) we conclude that Φ does not contain b explicitly. Thus,

$$a^{\theta}R = \Phi(a, P_2, P_3, P_4, \ldots . P_j),$$

and the theorem is established for reciprocants.

The Protomorphs for Reciprocants as far as extent 8 are as follows:

$P_2 = 4ac - 5b^2,$

$P_3 = a^2d - 3abc + 2b^3,$

$P_4 = 50a^2e - 175abd + 28ac^2 + 105b^2c,$

$P_5 = 10a^3f - 40a^2be - 12a^2cd + 65ab^2d + 16abc^2 - 39b^3c,$

$P_6 = 14a^2g - 63abf - 1350ace + 1782b^2e + 1470ad^2 - 4158bcd + 2310c^3,$

$$\begin{aligned} P_7 = {} & 7a^3h - 35a^2bg - 539a^2cf + 735ab^2f + 605a^2de + 306abce - 1485b^3e \\ & - 2135abd^2 + 1001ac^2d + 3465b^2cd - 1925bc^3, \end{aligned}$$

$$\begin{aligned} P_8 = {} & 420a^3i - 2310a^2bh - 25648a^2cg + 9240a^2df + 21780a^2e^2 + 36680ab^2g \\ & + 85386abcf - 191730abde - 59220ac^2e + 120540acd^2 \\ & - 126945b^3f + 252126b^2ce + 169260b^2d^2 - 419034bc^2d \\ & + 129360c^4 \end{aligned}$$

The work necessary for obtaining the first four of these, P_2, P_3, P_4, P_5, has been fully set out. Since P_4 is of degree 3, its second educt, G^2P_4, is of degree 5 and its reduced second educt of degree 4. A linear combination of this with a form whose leading term is a^2ce becomes divisible by a and gives P_6; but as this requires the preliminary calculation of the form (a^2ce), it is simpler to find P_6 directly by the method of indeterminate coefficients, and thence by eduction to get P_7 and P_8. Thus (to a numerical factor près) P_7 is the educt and P_8 the reduced second educt of P_6. Beyond this point the calculation of protomorphs has not at present been carried.

Referring to the table which gives the minimum degree and multiplicity for a Protomorph of any extent, we see that the multiplicity exceeds unity when the extent $j => 8$, and is exactly equal to 2 when $j = 8, 11$, or 21.

Hence the protomorphs as far as P_7 inclusive are unique; but there are two forms of extent 8 and degree 4 any linear combination of which (provided it contains the term a^3i) may be regarded as a protomorph. One of these forms is P_8, whose value is given above; the other is a linear combination of P_8 with a form, whose leading term is a^2cg, hereafter to be set forth.

The irreducible forms for extent 2 are a and P_2; every other form must be simply a power of P_2 multiplied by a power of a. We proceed to the calculation of all the Irreducible Forms for the extents 3 and 4 respectively. When $j = 3$, we may combine the protomorphs

$$4ac - 5b^2$$

and

$$a^2d - 3abc + 2b^3$$

with one another.

Adding 125 times the square of the latter to 4 times the cube of the former and dividing by a, there results the form

$$125a^3d^2 - 750a^2bcd + 500b^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c.$$

This form is analogous to the discriminant of the cubic, but is of a higher degree by one unit. Its type is 6; 5, 3, whereas that of the discriminant is 6; 4, 3.

In the case of invariants, we have to combine $ac - b^2$ with $a^2d - 3abc + 2b^3$. The square of the second, added to 4 times the cube of the first, gives

$$a^4d^2 - 6a^3bcd + 4a^2b^3d + 9a^2b^2c^2 - 12ab^4c + 4b^6 + 4a^3c^3 - 12a^2b^2c^2 + 12ab^4c - 4b^6.$$

Here the term $12ab^4c$ is nullified by $-12ab^4c$, so that the result contains a^2, the other factor being the discriminant

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2,$$

which is of the type 6; 4, 3.

We may show *à priori*, assuming the problematical but highly probable formula $(w;\ i, j) - (w-1;\ i+1, j)$, that the type 6; 4, 3 does not belong to any reciprocant.

For, as seen in the partitionments set out below,

$$(6;\ 4,\ 3) - (5;\ 5,\ 3) = 5 - 5 = 0$$

3.3	3.2
3.2.1	3.1.1
3.1.1.1	2.2.1
2.2.2	2.1.1.1
2.2.1.1	1.1.1.1.1

We can by no other means combine the protomorphs with one another or with the Quasi-Discriminant $(125a^3d^2\ldots.)$ so as to obtain additional fundamental forms. Every Rational Integral Pure Reciprocant of extent 3 is therefore necessarily a rational integral function of the four forms

deg. wt.
1.0 a,
2.2 $4M = 4ac - 5b^2$,
3.3 $A = a^2d - 3abc + 2b^3$,
5.6 $(a^3d^2) = 125a^3d^2 - 750a^2bcd + 500b^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c.$

These are connected by a syzygy of degree-weight 6.6, viz.:

$$125A^2 + 256M^3 = a\,(a^3d^2),$$

analogous to the syzygy of the same degree-weight, in the Theory of the Binary Cubic, which connects the Discriminant with a and the Protomorphs of extent 2 and 3.

It will be clearly seen from an inspection of the fundamental forms that there is no law for the coefficients of Reciprocants akin to that of their algebraical sum being zero in Invariants.

(*To be continued.*)